Announcements

1) HW #6 is now 60 points and has 3 more problems.

Definition: (normal)

$A \in M_n(\mathbb{C})$ is called **normal** if

$$\boxed{A^*A = AA^*}$$

Example : (normal / not normal)

$e_{i,j} \in M_n(\mathbb{C})$ is normal if and only if $i = j$. $(1 \leq i, j \leq n)$

$$e_{i,j}^* e_{i,j} = e_{i,j}^t e_{i,j}$$

$$= e_{j,i} e_{i,j} = e_{j,j}$$

Similarly,

$$e_{i,j} e_{i,j}^* = e_{i,i}$$

Also

$$A = \begin{bmatrix} 2 & i \\ -i & 3 \end{bmatrix}$$ is normal:

$A^* = A$, so

$$A^*A = A^2 = AA^*.$$

Lemma: Let $A \in M_n(\mathbb{C})$, A normal. (usual $\|\cdot\|_2$)

i) $\|Ax\|_2 = \|A^*x\|_2 \quad \forall x \in \mathbb{C}^n$

ii) $x \in \mathbb{C}^n$ is an eigenvector for A if and only if x is an eigenvector for A^*

iii) If x is an eigenvector for A, then $A(\{x\}^\perp) \subseteq \{x\}^\perp$

proof:

i) $\|Ax\|_2^2 = \langle Ax, Ax \rangle$

$$= \langle x, A^*Ax \rangle$$

(properties of A^*)

$$= \langle x, AA^*x \rangle$$

(normality)

$$= \langle A^*x, A^*x \rangle$$

$$= \|A^*x\|_2^2,$$

so $\|Ax\|_2 = \|A^*x\|_2$.

ii) Let λ be an eigenvalue for A. Then

$\boxed{A - \lambda I_n \text{ is normal}}$!

$$(A-\lambda I_n)^*(A-\lambda I_n)$$

$$= (A^* - \bar{\lambda} I_n)(A - \lambda I_n)$$

$$= A^*A - \bar{\lambda}A - \lambda A^* + |\lambda|^2 I_n$$

$$= A^*A - \bar{\lambda}A - \lambda A^* + |\lambda|^2 I_n$$

$$= AA^* - \lambda A^* - \bar{\lambda}A + |\lambda|^2 I_n$$

(normality of A)

$$= (A - \lambda I_n)(A^* - \bar{\lambda} I_n)$$

$$= (A - \lambda I_n)(A - \lambda I_n)^*$$

Now let x be an eigenvector for λ.

By i), since

x is an eigenvector

corresponding to $\gamma = 0$

for $A - \lambda I_n$,

$$0 = \| (A - \lambda I_n) x \|_2$$

$$= \| (A - \lambda I_n)^* x \|_2$$

$$= \| (A^* - \bar{\lambda} I_n) x \|_2$$

$\Rightarrow$ x is an eigenvector for A^*.

(corresponding to $\bar{\lambda}$)

iii) Let x be an eigenvector for A corresponding to λ. Let $y \in \{x\}^\perp$.

Then

$$\langle x, Ay \rangle$$

$$= \langle A^* x, y \rangle$$

$$= \langle \bar{\lambda} x, y \rangle \text{ (by ii)}$$

$$= \bar{\lambda} \langle x, y \rangle = 0.$$

$$\Rightarrow A(\{x\}^\perp) \subseteq \{x\}^\perp \quad \square$$

Theorem: Let $A \in \boxed{M_n(\mathbb{C})}$ be normal. Then A is diagonalizable.

proof: By induction.

If $n=1$, then A is multiplication by a scalar, which is a diagonal matrix.

If $n=2$, then by the fundamental theorem of algebra, A has an eigenvalue λ. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$ be an eigenvector

for A. Then

$$y = \begin{pmatrix} \overline{x_2} \\ -\overline{x_1} \end{pmatrix}$$ satisfies

$$\langle x, y \rangle = x_1 x_2 - x_2 x_1 = 0.$$

By the previous lemma,

$Ay \in \{x\}^{\perp}$.

But since $\mathbb{C}^2$ is 2-dimensional,

$Ay \in \text{span}\{y\} \Rightarrow$

$Ay = \gamma y$ for some $\gamma \in \mathbb{C}$.

Therefore, $\mathbb{C}^2$ has a basis consisting of eigenvectors of $A \Rightarrow A$ is diagonalizable.

(theorem from last class)

General n: Assume the result is true for $k = n-1$.

Again by the fundamental theorem of algebra, A has an eigenvalue λ with eigenvector x.

By our lemma,

A normal $\Rightarrow$

$$A(\{x\}^{\perp}) \subseteq \{x\}^{\perp}.$$

So with $W = \{x\}^{\perp}$

(isomorphic to $\mathbb{C}^{n-1}$),

by induction, $\exists$ a

basis of orthogonal

(eigen) vectors for W.

If we call this basis $\mathcal{B}_0$,

then $\mathcal{B} = \mathcal{B}_0 \cup \{x\}$ is

a basis for $\mathbb{C}^n$

$\Rightarrow$ A is diagonalizable

by theorem from last class. $\square$

Observe: (form of S)

The similarity matrix S is given by using the eigenvectors of A as its columns. If A is normal, we can choose the columns to be orthogonal from the previous proof. By normalizing, we may assume each vector has norm one.

Then

$$(S^*S)_{i,j}$$

$=$ inner product of eigenvector v_i with eigenvector v_j

$$= \delta_{i,j} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}.$$

$$\Rightarrow S^*S = I_n \Rightarrow S^* = S^{-1}$$
$$\Rightarrow SS^* = I_n$$

Special Cases

Definitions:

1) $U \in M_n(\mathbb{C})$ is said to be unitary if $\boxed{U^*U = UU^* = I_n}$

2) $A \in M_n(\mathbb{C})$ is said to be self-adjoint if $\boxed{A = A^*}$

Both 1) and 2) are families of normal matrices!

In particular, you can tell whether a matrix is self-adjoint by inspection; if it is, then you know it is diagonalizable!